MATH 2060 TUTO 12

§9.2

7. Discuss the series whose nth term is

(a) $\dfrac{n!}{3\cdot 5\cdot 7\cdots(2n+1)}$, (b) $\dfrac{(n!)^2}{(2n)!}$,

(c) $\dfrac{2\cdot 4\cdots(2n)}{3\cdot 5\cdots(2n+1)}$, (d) $\dfrac{2\cdot 4\cdots(2n)}{5\cdot 7\cdots(2n+3)}$.

Ans: a)

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!}{3\cdot 5\cdot 7\cdots(2n+1)(2n+3)}\cdot\frac{3\cdot 5\cdot 7\cdots(2n+1)}{n!}$$

$$= \frac{n+1}{2n+3}$$

$$\Rightarrow \lim\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{2}. < 1$$

By Ratio Test, the series $\sum a_n$ is absolutely convergent. //

c)

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2\cdot 4\cdots(2n)(2n+2)}{3\cdot 5\cdots(2n+1)(2n+3)}\cdot\frac{3\cdot 5\cdots(2n+1)}{2\cdot 4\cdots(2n)}$$

$$= \frac{2n+2}{2n+3} \longrightarrow 1$$ inconclusive from Ratio Test.

$$= 1 - \frac{1}{2n+3}$$ Try Raabe's Test

Then

$$\lim\left(n\left(1-\left|\frac{a_{n+1}}{a_n}\right|\right)\right) = \lim\left(\frac{n}{2n+3}\right)$$

$$= \frac{1}{2} < 1$$

By Corollary 9.2.9 (Raabe's test limiting form),

$\sum a_n$ is NOT absolutely convergent

$\Rightarrow \sum a_n$ is divergent since $a_n > 0 \quad \forall n$ //

§9.2

16. Let $\{n_1, n_2, \dots\}$ denote the collection of natural numbers that do not use the digit 6 in their decimal expansion. Show that $\sum 1/n_k$ converges to a number less than 80. If $\{m_1, m_2, \dots\}$ is the collection of numbers that end in 6, then $\sum 1/m_k$ diverges. If $\{p_1, p_2, \dots\}$ is the collection of numbers that do not end in 6, then $\sum 1/p_k$ diverges.

Ans: Let $A := \{n \in \mathbb{N} : n$ do not use digit 6 in their decimal expansion$\}$

Note $\# A \cap [1,9] = 8$

$\# A \cap [10,99] = 8 \times 9$ 1-9, no 6   0-9, no 6

$\# A \cap [100,999] = 8 \times 9 \times 9$

$\# A \cap [10^m, 10^{m+1}-1] = 8 \times 9^m \quad \forall m \geq 0$

So
$$\sum_{n_k < 10^{m+1}} 1/n_k = \sum_{i=0}^{m} \sum_{10^i \leq n_k < 10^{i+1}} \frac{1}{n_k}$$
$$\leq \sum_{i=0}^{m} 8 \times 9^i \cdot \frac{1}{10^i}$$
$$= 8 \sum_{i=0}^{m} \left(\frac{9}{10}\right)^i$$
$$\leq 8 \frac{1}{1-\frac{9}{10}} = 80 \quad \forall m \geq 0$$

By MCT, $\sum 1/n_k$ converges to a number ≤ 80.

$\sum 1/m_k = \sum_{n=0}^{\infty} \frac{1}{10n+6}$ which is divergent

since $\frac{1}{10n+6} \geq \frac{1}{16} \cdot \frac{1}{n} \quad \forall n \geq 1$

and the harmonic series $\sum 1/n$ is divergent.

$\{p_1, p_2, \dots\}$ includes every number that ends in 5.

So $\sum 1/p_k \geq \sum_{n=0}^{\infty} \frac{1}{10n+5}$

By comparison test again, $\sum 1/p_k$ is divergent. //

§ 9.3

6. Let $a_n \in \mathbb{R}$ for $n \in \mathbb{N}$ and let $p < q$. If the series $\sum a_n/n^p$ is convergent, show that the series $\sum a_n/n^q$ is also convergent.

Recall: Abel's Test

If • (x_n) is a convergent monotone seq

• $\sum y_n$ is convergent

then $\sum x_n y_n$ is also convergent.

Ans: Write $a_n/n^q = \left(\frac{1}{n^{q-p}}\right)(a_n/n^p)$

Note • $\frac{1}{n^{q-p}} \downarrow 0$ since $q > p$

• $\sum (a_n/n^p)$ is convergent.

By Abel's Test, $\sum (a_n/n^q)$ is also convergent //

9. If the partial sums of $\sum a_n$ are bounded, show that the series $\sum_{n=1}^{\infty} a_n e^{-nt}$ converges for $t > 0$.

Recall: Dirichlet's Test

If • $(x_n) \downarrow 0$

• partial sums of $\sum y_n$ are bounded

then $\sum x_n y_n$ is convergent.

Ans: Note • $\forall t > 0$, $e^{-nt} \downarrow 0$ as $n \to \infty$

• partial sums of $\sum a_n$ are bounded.

By Dirichlet's Test, $\sum a_n e^{-nt}$ is convergent $\forall t > 0$.

14. Show that if the partial sums s_n of the series $\sum_{k=1}^{\infty} a_k$ satisfy $|s_n| \leq Mn^r$ for some $r < 1$, then the series $\sum_{n=1}^{\infty} a_n/n$ converges.

Recall : Abel's Lemma

Let · (x_n), (y_n) be seq in $\mathbb{R}$, and

· $S_0 := 0$, $S_n := \sum_{k=1}^{n} y_k$.

Then, $\forall\, m > n$, $\sum_{k=n+1}^{m} x_k y_k = (x_m S_m - x_{n+1} S_n) + \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) S_k$

Ans: Apply Abel's Lemma to $x_n = \frac{1}{n}$ and $y_n = a_n$,

$\forall\, m > n$, $\sum_{k=n+1}^{m} \frac{a_k}{k} = \frac{1}{m} S_m - \frac{1}{n+1} S_n + \sum_{k=n+1}^{m-1} \left(\frac{1}{k} - \frac{1}{k+1}\right) S_k$

Since $|S_n| \leq M n^r$ $\forall n$, where $r < 1$, we have

$$\left| \sum_{k=n+1}^{m} \frac{a_k}{k} \right| \leq \frac{M m^r}{m} + \frac{M n^r}{n+1} + \sum_{k=n+1}^{m-1} \frac{M k^r}{k(k+1)}$$

$$\leq M m^{r-1} + M n^{r-1} + M \sum_{k=n+1}^{m-1} \frac{1}{k^{2-r}}$$

Now, 1) since $r-1 < 0$, $M m^{r-1}$, $M n^{r-1} \to 0$ as $m, n \to \infty$.

2) Since $2-r > 1$, the p-series $\sum \frac{1}{k^{2-r}}$ converges

Thus, $\forall\, \varepsilon > 0$, $\exists\, N \in \mathbb{N}$ s.t $\forall\, m > n \geq N$,

$$\left| \sum_{k=n+1}^{m} \frac{a_k}{k} \right| < \varepsilon + \varepsilon + \varepsilon = 3\varepsilon$$

By Cauchy criterion, the series $\sum_{n=1}^{\infty} \frac{a_n}{n}$ is convergent. //

§9.4

1. Discuss the convergence and the uniform convergence of the series $\sum f_n$, where $f_n(x)$ is given by:

(a) $(x^2+n^2)^{-1}$,
(b) $(nx)^{-2}$ $(x \neq 0)$,
(c) $\sin(x/n^2)$,
(d) $(x^n+1)^{-1}$ $(x \neq 0)$,
(e) $x^n/(x^n+1)$ $(x \geq 0)$,
(f) $(-1)^n(n+x)^{-1}$ $(x \geq 0)$.

Ans: e) 1) Note that

$$\lim f_n(x) = \begin{cases} 0 & , \ 0 \le x < 1 \\ \frac{1}{2} & , \ x = 1 \\ 1 & , \ x > 1 \end{cases}$$

So $\sum f_n(x)$ diverges on $[1, \infty)$

2) Fix $x \in [0,1)$.
Then $0 \le f_n(x) \le x^n$
Since the geometric series $\sum x^n$ converges,
$\sum f_n(x)$ converges also by comparison test

3) For $0 < a < 1$,
$0 \le f_n(x) \le x^n \le a^n \quad \forall x \in [0,a]$
Since $\sum a^n$ converges, Weierstrass M-test implies that
$\sum f_n$ converges uniformly on $[0,a]$.

4) $\sum f_n$ is not uniformly convergent on $[0,1)$:
$\forall n \in \mathbb{N}$, let $x_n = \frac{1}{2^{1/n}} \in [0,1)$.
Then, $\forall m > n$,

$$\left| \sum_{k=n+1}^{m} f_k(x_m) \right| \ge f_m(x_m) = \frac{\frac{1}{2}}{\frac{1}{2}+1} = \frac{1}{3}$$

By Cauchy Criterion 9.4.5,
$\sum f_n$ is NOT uniformly convergent on $[0,1)$ //

6. Determine the radius of convergence of the series $\sum a_n x^n$, where a_n is given by:

(a) $1/n^n$, (b) $n^\alpha/n!$,

(c) $n^n/n!$, (d) $(\ln n)^{-1}$, $n \geq 2$,

(e) $(n!)^2/(2n)!$, (f) $n^{-\sqrt{n}}$.

Ans: Set $\rho := \limsup(|a_n|^{1/n})$.

Then the radius of convergence R of $\sum a_n x^n$ is

$$R = \begin{cases} 0 & \text{if } \rho = +\infty \\ 1/\rho & \text{if } 0 < \rho < +\infty \\ +\infty & \text{if } \rho = 0 \end{cases}$$

d) Note that

$$1 = \ln e \leq \ln n \leq n \quad \forall n \geq 3$$

$$\Rightarrow \quad 1 \leq (\ln n)^{\frac{1}{n}} \leq n^{\frac{1}{n}} \quad \forall n \geq 3$$

Since $\lim n^{1/n} = 1$, it follows from Squeeze Thm that

$$\rho := \lim \left|\frac{1}{\ln n}\right|^{1/n} = 1$$

So radius of convergence is $R = 1/1 = 1$ //

f) $\forall n$, $\quad |n^{-\sqrt{n}}|^{1/n} = n^{-\frac{1}{\sqrt{n}}} = \left(\frac{1}{(\sqrt{n})^{\frac{1}{\sqrt{n}}}}\right)^2$

Note that $\lim_{x\to\infty} x^{\frac{1}{x}} = 1$

$\left(\text{since } \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0 \text{ and } x^{\frac{1}{x}} = \exp\left(\frac{1}{x}\ln x\right) \right)$

Thus $\rho = \lim |n^{-\sqrt{n}}|^{1/n} = \left(\frac{1}{1}\right)^2 = 1$

So radius of convergence is $R = 1/1 = 1$ //